Generalized coupled photon transport equations for handling correlated photon streams with distinct frequencies

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A generalized form of coupled photon transport equations that can handle correlated light beams with distinct frequencies is introduced. The derivation is based on the principle of energy conservation. For a single frequency, the current formulation reduces to a standard photon transport equation, and for fluorescence and phosphorescence, the diffusion models derived from the proposed photon transport model match for homogenous media. The generalized photon transport model is extended to handle wideband inputs in the frequency domain. © 2012 Optical Society of America

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The study of light propagation through a turbid medium that involves the coupling of multiple frequencies has a number of applications in many different disciplines, such as fluorescence spectroscopy [1], phosphorescence imaging [2], and laser Doppler flowmetry [3]. In these applications part of the incident light will change frequency inside the medium due to absorption and reemission by specific optical probes, as is the case in fluorescence and phosphorescence, or due to the Doppler effect resulting from scattering by moving particles in the medium, as is the case in laser Doppler flowmetry. Light propagation through a turbid medium is best described by the photon transport theory [4–6].

In this Letter we introduce a generalized photon transport equation involving a frequency change. In our previous work we have specifically developed this theory for fluorescence [7] and phosphorescence [8]. For both these cases we have showed that the systematic derivation of the diffusion approximation starting from the more accurate photon transport model resulted in one additional diffusion term that had been omitted in the conventional diffusion models. Using realistic optical parameter values, we showed that the omission of this new term in both fluorescence spectroscopy and phosphorescence imaging of biological tissue might lead to significant errors in inhomogeneous media. The main focus of this Letter is to generalize this idea and to show that it can be extended to other scenarios such as laser Doppler flowmetry.

Figure 1 shows a schematic diagram of possible paths traced by photons in a turbid medium having both elastic and inelastic scatterers. Inelastic scatterers result in scattered photons of a frequency different from incident photons, while elastic scatterers scatter without a frequency change. Figure 2 shows a schematic diagram of the transfer of photon energy through an infinitesimal volume element. We assume that this volume element receives radiation at r with intensities (radiiances) $I_{1,in}$ and $I_{2,in}$ corresponding to frequencies $\nu_1$ and $\nu_2$, respectively, and passes through an element of length $\Delta s$ and cross section $\Delta A$ normal to the direction of the ray $\Omega$ into the solid angle $\Delta \Omega$ in time $\Delta t$ in the corresponding frequency intervals $\Delta \nu_1$ and $\Delta \nu_2$. Here, we assume that $\nu_1$ and $\nu_2$ have finite linewidths, which are characterized by $\Delta \nu_1$ and $\Delta \nu_2$. Let the intensities of the radiation corresponding to frequencies $\nu_1$ and $\nu_2$ emerging at $r+\Delta r$ at the end of time $t+\Delta t$ be $I_{1,out}$ and $I_{2,out}$, respectively. Let the number of incident photons per unit solid angle per unit area to this infinitesimal volume of frequencies $\nu_1$ and $\nu_2$ be given by $G(I_{1,in})$ and $G(I_{2,in})$, respectively, and the number of emerging photons per unit solid angle per unit area be given by $G(I_{1,out})$ and $G(I_{2,out})$, respectively. Let the number of photons of $\nu_1$ lost per unit solid angle per unit area due to a frequency change from $\nu_1$ to $\nu_2$ as a result of inelastic scattering be given by $G_{L,\nu}(I_{1,in},I_{2,out})$ and the number of photons of $\nu_2$ gained per unit solid angle per unit area due to a frequency change from $\nu_1$ to $\nu_2$ as a result of inelastic scattering be given by $G_{G,\nu}(I_{1,in},I_{2,out})$; let the number of photons lost due to absorption and scattering from the direction $\Omega$ into other directions be $G_{\text{loss}}$ and the number of photons gained due to scattering from other directions into the direction $\Omega$ be $G_{\text{gain}}$. Then, the conservation of the number of photons results in

![Fig. 1. (Color online) Schematic diagram of possible paths traced by photons in a turbid medium having both elastic and inelastic scatterers.](image-url)
the amount of photon energy at frequency $\nu_1$ lost due to coupling. We have assumed that the frequency change occurs from $\nu_1$ to $\nu_2$ only. Hence, the coupling functions in both equations are functions of $I_{1,in}$. However,

$$I_m(r + \Delta r, \Omega, t + \Delta t) - I_m(r, \Omega, t) = \left( \frac{1}{c} \frac{\partial}{\partial t} + \Omega \cdot \nabla \right) I_m(r, \Omega, t) ds + O(ds),$$

where $m = 1, 2$ and $c$ is the speed of light in the medium [9]. Hence, Eqs. (3) and (4) can be written as

$$\left( \frac{1}{c} \frac{\partial}{\partial t} + \Omega \cdot \nabla \right) I_2(r, \Omega, t) = H_{G,couple}(I_1(r, \Omega, t))$$

$$- \sigma^{(2)}_a I_2(r, \Omega, t) + \sigma^{(2)}_s \int_{4\pi} P_2(\Omega'; \Omega') I_2(r, \Omega', t) \frac{d\Omega'}{4\pi},$$

$$- \sigma^{(1)}_a I_1(r, \Omega, t) + \sigma^{(1)}_s \int_{4\pi} P_1(\Omega; \Omega) I_1(r, \Omega', t) \frac{d\Omega'}{4\pi},$$

where $\sigma^{(m)}_a = \sigma^{(m)}_a + \sigma^{(m)}_s$ is the attenuation coefficient. It can be seen that when there is no coupling between the two frequencies of light, the two photon transport equations reduce to the standard photon transport equation corresponding to each frequency of light. Equations (6) and (7) are the generalizations of the coupled photon transport equations. The functions $H_{G,couple}$ and $H_{L,couple}$ take different forms for each application and are determined by the physical conditions of the system.

For example, in fluorescence spectroscopy, $H_{G,couple} = (1/\tau) \phi_{eff} \sigma^{(1)}_{a,ph}/(4\pi) \int_{4\pi} \int_0^{\infty} \exp[-(t - t')/\tau] I_1(r, \Omega, t') dt' \frac{d\Omega}{4\pi}$ and $H_{L,couple} = \sigma^{(1)}_{a,ph} I_1(r, \Omega, t)$, where $\tau$ is the fluorescence lifetime, $\phi_{eff}$ is the quantum efficiency, and $\sigma^{(1)}_{a,ph}$ is the absorption coefficient of fluorophores at $\nu_1$. In phosphorescence imaging, $H_{G,couple} = (\exp(-(t - t')/\tau) \phi_{eff} \sigma^{(1)}_{a,ph}/(4\pi) \int_{4\pi} \int_0^{\infty} I_1(r, \Omega, t') dt' \frac{d\Omega}{4\pi}$ and $H_{L,couple} = \sigma^{(1)}_{a,ph} I_1(r, \Omega, t)$, where $\sigma^{(1)}_{a,ph}$ is the absorption coefficient of phosphors at $\nu_1$.

In laser Doppler flowmetry we can introduce the coupling via a phase function [4] $P_{couple}(\Omega; \Omega')$ that gives the probability of a photon with a frequency $\nu_1$ in the direction $\Omega$’ being scattered into the direction $\Omega'$ with an apparent frequency (due to the Doppler effect) of $\nu_2$. Then we can write the coupling functions for this case as

$$H_{G,couple} = \sigma^{(1)}_{s,mov} \int_{4\pi} P_{couple}(\Omega; \Omega') I_1(r, \Omega', t) \frac{d\Omega'}{4\pi},$$

$$H_{L,couple} = \sigma^{(1)}_{s,mov} \int_{4\pi} P_{couple}(\Omega'; \Omega) I_1(r, \Omega, t) \frac{d\Omega'}{4\pi} = \sigma^{(1)}_{s,mov} I_1(r, \Omega, t).$$

We have considered only narrowband inputs thus far. To handle wideband inputs that contain a range of...
frequencies, it is easier to work in the frequency domain. We slice the incident spectrum and consider that it consists of \( N \) distinct frequencies, \( \nu_k \) to \( \nu_N \). Suppose the inelastic scatterers result in broadening of the frequency spectrum and that the output spectrum also consists of \( N \) distinct frequencies, \( \nu_k \) to \( \nu_N \). Then, the conservation of the number of photons results in

\[
G(I_{m,\text{out}})dAd\Omega = G(I_{m,\text{in}})dAd\Omega - G_{L,\text{couple}}(I_{m,\text{in}}, I_{1,\text{out}})dAd\Omega - \cdots - G_{L,\text{couple}}(I_{m,\text{in}}, I_{N,\text{out}})dAd\Omega - \cdots - G_{G,\text{couple}}(I_{1,\text{in}}, I_{m,\text{out}})dAd\Omega + \cdots + G_{G,\text{couple}}(I_{N,\text{in}}, I_{m,\text{out}})dAd\Omega + \cdots - G_{\text{loss}}(I_{1,\text{in}})dAd\Omega + G_{\text{gain}}(I_{1,\text{in}})dAd\Omega,
\]

(10)

where \( m = 1, \ldots, N \). Here, we have considered the most general case, where all the input frequencies \( \nu_k \) to \( \nu_N \) couple to each other. As for the narrowband case discussed above, we can write Eq. (10) in terms of radiance, and then by applying the Fourier transform

\[
F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt,
\]

we can write Eq. (10) in the frequency domain as

\[
\left(\frac{j\omega m}{c} + \nabla\cdot \nabla\right)I(r, \Omega, \omega_m) = -\mathcal{L} \omega_m^{m\rightarrow n} I(r, \Omega, \omega_m) - \cdots - \mathcal{L} \omega_1^{m\rightarrow n} I(r, \Omega, \omega_m) + \mathcal{L} \omega_0^{m\rightarrow 0} I(r, \Omega, \omega_1) + \cdots + \mathcal{L} \omega_0^{m\rightarrow 0} I(r, \Omega, \omega_N) + \cdots + \mathcal{L} \omega_0^{N\rightarrow m} I(r, \Omega, \omega_N) - \sigma_i(m) I(r, \Omega, \omega_m) + \sigma_s(m) \int_{4\pi} P_m(\Omega, \Omega') I(r, \Omega', \omega_m) d\Omega',
\]

(12)

where \( m = 1, \ldots, N \) and \( m \neq n \). \( \mathcal{L} \omega_m^{m\rightarrow n} \) is the operator that operates on \( I(r, \Omega, \omega_m) \) to account for the amount of photon energy lost due to coupling from \( \nu_m \) to \( \nu_n \), and \( \mathcal{L} \omega_0^{m\rightarrow 0} \) is the operator that operates on \( I(r, \Omega, \omega_m) \) to account for the amount of photon energy gained due to coupling from \( \nu_k \) to \( \nu_m \). Here, we have assumed that the operators \( \mathcal{L} \omega_m^{m\rightarrow n} \) and \( \mathcal{L} \omega_0^{m\rightarrow 0} \) are linear, which is the case for almost all the scenarios we might encounter.

We can write the coupled set of equations in Eq. (12) in matrix format as

\[
\Omega \cdot \nabla I(r, \Omega, \omega) + WI(r, \Omega, \omega) = \mathcal{L}I(r, \Omega, \omega) + \Gamma I(r, \Omega, \omega) + \frac{1}{4\pi} \int_4 P(r, \Omega', \omega) d\Omega',
\]

(13)

where \( I(r, \Omega, \omega) \) and \( I(r, \Omega', \omega) \) are \( N \times 1 \) column vectors, \( \mathcal{L} \) is an \( N \times N \) matrix, and \( W, \Gamma, \) and \( P \) are \( N \times N \) diagonal matrices. \( I_m(r, \Omega, \omega) = I(r, \Omega, \omega_m) \), \( I_m(r, \Omega, \omega) = I(r, \Omega, \omega_m) \), \( W_{mn} = j\omega_m/c \), \( \Gamma_{mn} = \sigma_i(m) \), \( P_{mn} = \sigma_s(m) P_m(\Omega, \Omega) \), and

\[
\mathcal{L}_{mn} = \left\{ \begin{array}{ll}
-\sum_{k=1}^N \mathcal{L} \omega_k^{n\rightarrow k} & m = n; m \neq k \\
\mathcal{L} \omega_0^{m\rightarrow n} & m \neq n
\end{array} \right.
\]

With the matrix formulation given by Eq. (13), it will be easier to handle and solve problems that deal with frequency changes and wideband inputs.

Most widely used methods for modeling photon transport involving frequency changes are the diffusion approximation [1,10] and the Monte Carlo method [10,11]. The proposed model does not have an analytical solution, as opposed to the diffusion approximation, and it is comparatively more difficult to solve numerically. For optically thick highly scattering media with comparatively less absorption, the diffusion approximation could be advantageous. However, for optically thin media with sparse fluorophore concentrations, and when absorption dominates over scattering, the proposed model produces more accurate results. Most significantly, the proposed model can be solved using spectral expansions, finite differences, and discrete ordinates and produces results much faster than the computationally intense Monte Carlo method, which is inevitably affected by statistical errors due to finite sampling space. However, the extension of the proposed model for handling complex geometries and inhomogeneous properties will not be as flexible as the Monte Carlo method.

References